

Optimal Control Model of Software Quality for Digital Vendors

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Abstract

This paper models a firm's level of effort in mobile software application production as an optimal control problem to maximize market share and profit in a digital marketplace. Through a rumor-spreading system dynamic, a digital goods vendor can capture market share by focusing efforts on software quality, which is costly. We first derive theoretical results expressing market share, the state variable, in terms of effort put towards quality, the control variable. We then use numerical examples to illustrate optimal control paths for vendors facing various market conditions. Our results on effort towards quality are consistent with software evolution theory and show that focus on quality always declines over a product's life cycle. On the other hand, our findings on market share indicate that initial market conditions dictate the optimal path to pursue and affect final market share.

1 Introduction

Digital distribution platforms (e.g., GooglePlay, iTunes, Steam) have revolutionized the way consumers purchase digital products like music, software applications (apps), and video games. These platforms significantly decrease the time and startup costs required to market digital goods, facilitating the distribution of these products to consumers anywhere in the world. By eliminating the physical barriers to entry, digital distribution platforms have redefined the software supply chain to no longer necessitate brick and mortar manufacturers, distribution centers, and storefronts.

This paradigm shift in product distribution is most prominent in software applications for mobile devices, or mobile apps. These products include games, social media, and utility applications for all manners of mobile devices; smart phones, tablets, smart watches, and even eyewear in the form of Google Glasses can run mobile apps. A unique feature of mobile apps is that they are almost exclusively distributed digitally, where individual software developers rely on established digital distribution platforms, such as Apple's App Store or Google's GooglePlay, for marketing. Consequently, software developers can choose to focus their efforts purely towards app development, while companies like Apple and Google look to market the myriad of new software on their respective digital distribution platforms. These platforms represent a significant venue of profit; for example, Apple's App Store, recorded over \$10 billion in sales during 2013 alone [1].

The digital markets for mobile apps represent a new, unstudied environment for product distribution, one where developers and distributors must cooperate in the face of new strategic challenges. Large distributors like Apple want to ensure only high-quality products enter the digital market, and the most common implementation of quality-assurance is user-generated reviews. User reviews allow consumers to interact with one another and influence the reputation and desirability of a given digital good. In addition, these reviews foster rumor-spreading/viral propagation marketing

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within their platforms, which we aim to capture in our model. In turn, firms in the software industry hope to rely on reputation and past successes to retain market share. On the other hand, even the most entrenched corporations can be unseated if quality suffers and alternatives are available. A decreased effort towards quality is eventually realized in the market, and consumers react accordingly. Likewise, a strong focus on quality is also recognized; however, maintaining this dedication is expensive and faces decreasing returns over time. Therefore, developers must balance quality, cost, and market share capture to maximize profits for a product’s life cycle.

In this paper, we model the strategic decision-making process that app developers face as an optimal control problem. Firms choose their level of effort towards quality over a product life cycle to capture market share and maximize total revenue stream for a digital good. First we introduce our control model of a single vendor’s optimal path for focus on quality and resulting market share. From our model, we prove analytical results in this environment and establish the basic interaction between market share and quality in a dynamic setting. We then show that our techniques can be applied to a more general theoretical framework, providing researchers with a new approach to solving optimal control problems in similar environments. Next, we illustrate the model’s applications using numerical examples and highlight insights for firms. Our theoretical and numerical findings demonstrate that, in the face of quadratic costs on quality dedication, firms should steadily lower their focus on quality over any product’s life cycle. These results are consistent with the established literature on software evolution, specifically Lehman’s 7th law of software evolution (see [2]).

1.1 Purpose and Organization of this Paper

This paper advances the literature in the following ways: (i) We derive a novel model of choice with respect to effort towards quality that qualitatively explains decisions made by mobile app developers. (ii) To explore the theoretical properties of our model, we exploit a special structure of the optimal control problem and illustrate a novel method of analysis. (iii) We show that a commonly accepted law of software engineering, i.e., Lehman’s 7th law of software evolution, emerges as a natural property of our control model.

The remainder of this paper is organized as follows: In Section 2 we review the relevant literature. In Section 3 we present an optimal control model where a single firm optimizes their effort towards quality to capture market share. We then generalize our findings to a broader class of optimal control problems and provide a novel approach to solving these systems of differential equations. Theoretical results are presented in this section, and insights for digital distributors are discussed. In Section 4 we provide numerical examples highlighting the qualitative behavior of the model under various market conditions. Conclusions are presented in Section 5.

2 Related Literature

Digital distribution has been a popular topic of research in information systems and management science in the past two decades. Within the literature, Bakos and Brynjolfsson looked at marketing digital goods in bundles as a pricing strategy [3]. The authors demonstrate among other things that, due to the near-zero marginal cost characteristic of digital goods, bundling of goods eases consumers’ valuations on the quality of these products. In 2001, Bhargava and Choudhary analyze vertical differentiation of as a marketing strategy for digital goods from the single vendor monopolist’s perspective. The authors show that it is optimal to offer a single quality product to all customers rather than utilizing second-degree price discrimination to allow customers to self-select [4]. [5] continue this analysis in Management Science by studying differences in cost structure

and competition between digital goods and industrial goods. The authors find that markets for information goods lack the segmentation inherent in markets for traditional goods. A key result is that a monopolistic firm will offer only a single digital product. Competition can lead to highly concentrated information-good markets, though the leading firm is able to behave like a monopolist even with no barriers to entry. We utilize these concepts to motivate our model for finding the optimal control path for quality for single vendors.

The quality and time-to-market trade-offs have been extensively analyzed for physical products. By positioning quality as the key variable in a firm’s decision-making, scholars have explored the implications of quality focus and its impacts on revenue in traditional markets. Factors such as speed to market, strength of competition, and development costs are all crucial factors in this decision process [6]. [7] extends research on the quality of physical goods to multiple generations of production. The authors find that more total time is devoted to product development under their multiple generation model, leading to more, higher quality products over time. [8] model the strategic choice of quality as a two-firm, two-staged game and showed the importance of focusing on quality to profit. In both simultaneous and sequential games, choosing the higher level of quality captures greater market share and profits.

Focus on quality for digital goods is important as well; however, the established literature on software evolution notes that the quality of a digital good or system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes [2]. This theory is evaluated by Yu and Mishra’s empirical work on software products, which uses the accumulated defects-density metric to measure the focus on quality [9]. The authors apply this metric to two open source software programs and their bug reports, which are data mined. Their findings supports Lehman’s 7th law, noting that focus on quality for software does indeed decrease over the product’s lifespan in the way of increasing defect reports.

Correctly pricing digital products is another unique challenge vendors in the market face. Though we do not explicitly cover pricing in our paper, our model implicitly link effort towards quality to higher prices through the profit function. [10] analyzed the optimal pricing of such goods in an environment with incomplete information. The author highlights the various pricing schemes available to firms and scenarios where they are best utilized, such as fixed-fee in emerging markets and non-linear usage-based fees in saturated markets. [11] extend this line of research with their paper in 2011 by studying digital products with discontinuous cost functions. For industries with this structure, the paper demonstrates that a full cost recovery IT chargeback system is not optimal under many circumstances a firm may face. [12] add to the pricing literature by considering a digital goods monopolist with production costs that are quadratic in quality facing piracy concerns. The authors note that, assuming the vendor offers a single quality level to maximize profits, lower piracy enforcement can increase the firm’s incentives to focus on quality under certain conditions.

3 Optimal Control Model of Digital Distribution

In this section, we first construct our optimal control model for a single vendor in a mobile apps market. We use this analytic framework to establish the unique relationship between the state, market share, and the control, effort towards quality. We then generalize our findings to any optimal control problem with the same underlying structure for the equation of motion. Lastly, we consider the scenario where marginal revenue does not depend on effort towards quality, which is relevant in certain digital markets.

3.1 Digital Distribution Model

Our goal is to model the behavior of vendors using digital distribution platforms to market their mobile app. We begin with a single firm looking to maximize profits over a product's life cycle by controlling its effort towards quality. Therefore, the firm aims to solve the following maximization problem:

$$\Psi(x(T)) + \int_0^T R(u_t)x - Cu_t^2 dt \quad (1)$$

Let $u_t : \mathbb{R}_+ \rightarrow [0, 1]$ be a single valued function of time that captures all efforts related to generating a quality product. The efforts made towards the design of the product could be: the debugging process, on-going customer support, content creation, or even marketing expenditure. It is important to note that maintaining effort towards quality for digital goods is costly, which is reflected in our profit function. Therefore, u_t denotes the focus on quality at time t , where $u_t \in [0, 1]$, with 0 representing very little concern for quality and 1 representing almost complete dedication towards quality. Let x_t denote the proportion of users who have adopted the vendor's product at time t , i.e., the market share at time t .

This functional form states that the revenue stream of the firm depends on both u_t ¹ and x_t . By assuming the revenue stream depends on u_t , we are implicitly stating that consumers recognize quality products and are willing to pay more for them, now and in the future. This assumption has merit in digital markets for mobile gaming apps; games are priced in tiers, and the mobile games with highest perceived quality can debut for \$5 to \$10. Games created by less well-known companies or independent developers normally sell for much less, often 0 price, as the perceived quality is lower. These free-to-play games comprise a large proportion of the mobile market, as their ease of access encourages users to adopt the mobile app at very little cost. Consumers are then enticed to spend more money when developers dedicate effort towards creating quality content.

Costs in the profit function are quadratic² and depend only on u_t . $\Psi(x(T))$ is the salvage value of market share at time T and constitutes the transversality condition. We take $R(u)$ to be a linear function such that $R(u) = \alpha u$. To motivate our equation of motion, we assume a population of users who will choose to purchase the vendor's product if the perceived quality of the product surpasses the utility gained from using the default product. Users are assumed to be homogeneous and judge a product based solely on perceived quality. We model the dynamics of rumor spreading and its impact on market share in our equation of motion:

$$\dot{x} = \beta \cdot x(1 - x) \cdot (\pi(u) - \pi_0) \quad (2)$$

The relationship between $\pi(u)$ and π_0 governs the sign of the equation of motion. $\pi(u)$ is the value function of u , which denotes the utility users derive from the product at time t . π_0 is the value users derive from the next best product, assumed to be constant. For simplicity, we can normalize $\pi(u)$ and π_0 to lie between 0 and 1 as well without losing analytic power. When $\pi(u) > \pi_0$, indicating that the value derived from the firm's digital product is greater than the value of the next best alternative, the sign of \dot{x} is positive, and market share increases. The reverse holds as well. Henceforth, let $\pi(u) = P \cdot u$, which is linear and monotonically increasing.

The equation $\dot{x} = \beta \cdot x(1 - x)$ is a logistic model of rumor spreading, and the intuition behind the model is straightforward. Market share changes via rumor spreading, whether positive or negative, are greatest when half of the population owns the product. When market share is small (close to

¹The assumption that revenue depends on u_t is a crucial one and dictates how the system moves through time. We will analyze how this assumption impacts the results.

²This is in the same vein as Lahiri & Dey 2013.

0), rumors about the mobile app spread slowly because only a small proportion of the population own the product. When market share is large (close to 1), rumors spread slowly because most of the population have already adopted the product.

Finally, we let $\Psi(x(T)) = \rho \cdot x(T)$, with $\rho \in [0, 1]$, so that our salvage value function is linear. Our optimal control problem becomes:

$$\begin{cases} \max & \rho x(T) + \int_0^T \alpha x_t u_t - C u_t^2 \, dt \\ \text{s.t.} & \dot{x} = \beta \cdot x(1-x) \cdot (Pu - \pi_0) \\ & x(0) = x_0 \\ & u \in [0, 1] \end{cases} \quad (3)$$

From (3), the standard optimal control approach (see [13]) is to numerically solve the system using the two-point boundary-value problem, which we do in Section 4. However, we can first exploit the structure of the model to solve for \dot{u} . This non-standard technique allows us to glean insight into the relationship between market share and effort towards quality over time.

Proposition 3.1. *Let $u \in [0, 1]$. Then x and u satisfy the following differential equations:*

$$\begin{cases} \dot{x} = \beta \cdot x(1-x) \cdot (Pu - \pi_0) \\ \dot{u} = -\frac{\alpha\beta\pi_0}{2C} \cdot x(1-x) \end{cases} \quad (4)$$

Proof. Proof of Proposition 3.1: If $\lambda(t)$ is the co-state, then the Hamiltonian is given as:

$$H(x, u, \lambda, t) = \alpha \cdot u_t \cdot x_t - C \cdot u_t^2 + \lambda_t [\beta \cdot x_t \cdot (1 - x_t) \cdot (P \cdot u_t - \pi_0)] \quad (5)$$

Optimizing with respect to u_t yields:

$$u_t = \frac{-x_t(-\alpha - \lambda_t \beta P + \lambda_t \beta x_t P)}{2C} \quad (6)$$

Using the Pontryagin conditions, we solve for $\dot{\lambda}$, which is omitted here. \dot{x} is already given, so we can now differentiate equation 6 using the derived values for $\dot{\lambda}$ and \dot{x} . This gives us the expression above for \dot{u} purely as a function of the state. \square \square

Note that $\dot{u} < 0$, so that at optimality, focus on quality is a monotonically decreasing function. This is consistent with Lehman's 7th law of software evolution, which states that the quality of a digital good or system will appear to be declining unless it is rigorously maintained and adapted to operational environment changes [2]. Because focus on quality is expensive to maintain, we see that as a product approaches the end of its life cycle, the perceived quality is declining through time. Consequently, Lehman's 7th law of software evolution is an emergent property of this model. From Proposition 3.1, we can express x solely as a function of u to illustrate the cases in which app developers may find themselves while on the optimal control path.

$$\frac{dx}{du} = -\frac{2C}{\alpha\pi_0} (Pu - \pi_0) \quad (7)$$

Thus,

$$x(u) = \tilde{C} - \frac{CP}{\alpha\pi_0} u^2 + \frac{2C}{\alpha} u \quad (8)$$

where \tilde{C} is the constant of integration. With this expression, we have computed an expression for x in terms of u and see that x varies quadratically in u . From Equation 4, we see that $u(t)$ is a monotonically decreasing function of time. Therefore $u(T) - u(0) \leq 0$. Furthermore, we know that $x(u)$ is maximized when $u = \pi_0/P$, which gives us the following result:

Proposition 3.2. *Exactly one of the following holds:*

1. $u(0) > \pi_0/P$, $u(T) \geq \pi_0/P$ and thus $x(T) \geq x(0)$ and market share increases monotonically.
2. $u(0) > \pi_0/P$, $u(T) < \pi_0/P$ and market share increases and then decreases, and the final relationship between $x(T)$ and $x(0)$ is determined by the relationship between $u(T)$ and $u(0)$.
3. $u(0) \leq \pi_0/P$ and thus $x(T) \leq x(0)$ and market share decreases monotonically.

Proposition 3.2 describes the inherent relationship between effort towards quality and market share in the model. Depending on the specific parameters for $x(0)$, π_0 , and P , a digital vendor may find itself in any of these scenarios. In Section 4, we explore the shape of the control path and its implications for firms.

Our next proposition describes the situation when a mobile app would experience monotonically decreasing market share over its product life cycle.

Proposition 3.3. *Market share decreases (i.e., $x(T) \leq x(0)$) if and only if:*

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 0 \quad (9)$$

Proof. Proof of Proposition 3.3: From Expression 8, we may compute:

$$\begin{aligned} x(T) - x(0) &= \frac{2C}{\alpha} (u(T) - u(0)) - \frac{CP}{\alpha\pi_0} (u(T)^2 - u(0)^2) = \\ &\quad \frac{2C}{\alpha} (u(T) - u(0)) - \frac{CP}{\alpha\pi_0} (u(T) + u(0)) (u(T) - u(0)) = \\ &\quad \left[\frac{2C}{\alpha} - \frac{CP}{\alpha\pi_0} (u(T) + u(0)) \right] (u(T) - u(0)) = \\ &\quad \frac{C}{\alpha} \left[2 - \frac{P}{\pi_0} (u(T) + u(0)) \right] (u(T) - u(0)) \quad (10) \end{aligned}$$

As noted, $u(T) - u(0) \leq 0$. Thus, $x(T) - x(0) \leq 0$ if and only if:

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 0 \quad (11)$$

This completes the proof. □

Corollary 3.4. *If $\pi_0 \geq P$, then $x(T) - x(0) \leq 0$.*

Proof. Proof of Corollary 3.4: From the proposition, we know that $x(T) - x(0) \leq 0$ if and only if:

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 0$$

Note, $u(T) + u(0) \leq 2$. Thus,

$$2 - \frac{P}{\pi_0} (u(T) + u(0)) \geq 2 - 2\frac{P}{\pi_0} \quad (12)$$

Thus:

$$2 - 2\frac{P}{\pi_0} \geq 0 \implies x(T) - x(0) \leq 0 \quad (13)$$

So:

$$\pi_0 \geq P \implies x(T) - x(0) \leq 0 \quad (14)$$

This completes the proof. \square \square

Proposition 3.3 and its corollary provides an obvious check on the sensibility of the model: if a user derives less utility from the product than the alternative, market share must (necessarily) decrease as the alternate is adopted. This holds as long as $u \in [0, 1]$.

Lastly we can relate \tilde{C} directly to $x(T)$. From Expression 26 for $u(T) \in [0, 1]$:

$$u(T) = \left[\frac{\alpha x(T) + \beta P x(T) (1 - x(T)) \rho}{2C} \right]_0^1 \quad (15)$$

Furthermore:

$$\tilde{C} = x(T) + \frac{CP}{\alpha \pi_0} u(T)^2 - \frac{2C}{\alpha} u(T) \quad (16)$$

From Equation 15, we can analyze the equation for optimal $u(T)$ in two parts. The first part, $\frac{\alpha x(T)}{2C}$, represents the marginal profit for the revenue stream; the second part, $\frac{\beta P x(T) (1 - x(T)) \rho}{2C}$, represents the marginal increase in market share. It is also worth noting from Equation 15 that if the marginal salvage value, $\Psi'(x(T)) = \rho$, is 0, then $u(T) = [\frac{\alpha x(T)}{2C}]_0^1$. We have the following result, which is a direct consequence of Expression (15).

Proposition 3.5. *The final value $u(T)$ is positive if and only if $x(T)$ is positive.*

Proof. Proof of Proposition 3.5: The expression:

$$\frac{\alpha x(T) + \beta P x(T) (1 - x(T)) \rho}{2C} \quad (17)$$

is concave in $x(T)$ and must be non-negative because of the signs of the parameters and the fact $x(T) \in [0, 1]$. It has zeros in $x(T)$ only when $x(T) = 0$ and when:

$$x(T) = \frac{\alpha + \beta P \rho}{\beta P \rho} \quad (18)$$

Note, the previous expression is always greater than 1 for any $\rho > 0$ and this root does not exist when $\rho = 0$. Thus, $u(T) > 0$ if and only if $x(T) > 0$. \square \square

From the propositions given in this section, we begin to comprehend the underlying structure of digital market share with rumor spreading. We confirm one of Lehman's empirical laws of software evolution as well as delineate the different optimal control paths of effort that developers should follow. In Section 4, we construct these control paths numerical. In the next subsection, we study the general properties of our model that make this analytic method possible. Before moving to a numerical analysis of example problems, we study a constant revenue variation of the original model.

3.2 Generalized Model

During our derivation of the digital distribution model and Proposition 3.1, we realized that the methodology of solving for \dot{u} in closed-form is non-standard in the optimal control literature. Therefore, in this section, we look to generalize our findings and make a methodological contribution to optimal control problems.

Consider the following optimal control problem:

$$\begin{cases} \max & \Psi(x(T)) + \int_0^T a(x) + b(x)u + ru^2 \, dt \\ \text{s.t.} & \dot{x} = f(x) + ug(x) \\ & x(0) = x_0 \end{cases} \quad (19)$$

For simplicity, we restrict our attention to a single state x and a single control u , with the knowledge that some results may be extended to the case when the state x is a vector. We also assume that $a, b, f, g : \mathbb{R} \rightarrow \mathbb{R}$ are \mathcal{L}^2 functions and $r \in \mathbb{R}$. Our model in section 3.1 adheres to this general form.

The Hamiltonian for this system is:

$$\mathcal{H}(x, u, \lambda) = a(x) + b(x)u + ru^2 + \lambda(f(x) + ug(x)) \quad (20)$$

When $r < 0$, the resulting (Hamiltonian maximizing) optimal control as a function of the state and co-state is given by:

$$u^*(x, \lambda) = \frac{-b(x) - g(x)\lambda}{2r} \quad (21)$$

Otherwise, the controller will choose extreme values for u^* , if some constraints of the form $u \in \mathcal{U}$ are provided. Let us focus on the case when u^* is governed by Equation 21.

We can also solve for $\dot{\lambda}$ by utilizing the Pontryagin & transversality conditions:

$$\dot{\lambda} = -\frac{d\mathcal{H}}{dx} = -a'(x) - u(b'(x) + \lambda g'(x)) - \lambda f'(x), \quad (22)$$

and

$$\lambda(T) = \Psi'(x(T)) \quad (23)$$

which can be written entirely in terms of x and λ by using Equation 21 (in the usual way) to obtain the two-point boundary value problem for x and λ .

From Equation 21 we can compute \dot{u} as well:

$$\dot{u} = \frac{1}{2r} \left(-b'(x)\dot{x} - g'(x)\dot{x}\lambda - \dot{\lambda}g(x) \right) \quad (24)$$

Substituting \dot{x} and $\dot{\lambda}$ from above, we obtain an expression for \dot{u} in terms of λ and x :

$$\dot{u} = \frac{g(x)a'(x) - f(x)b'(x)}{2r} + \lambda \left(\frac{g(x)f'(x) - f(x)g'(x)}{2r} \right) \quad (25)$$

Thus we have the following proposition:

Proposition 3.6. *If $g(x)f'(x) - f(x)g'(x) = 0$, then \dot{u} is solely a function of the state x . Furthermore, as long as $u(T) \in [0, 1]$, we have:*

$$u(T) = \frac{-b(x(T)) - g(x(T))\Psi'(x(T))}{2r} \quad (26)$$

Thus, we have a two point boundary value problem in only the state and control:

$$\begin{cases} \dot{x} = f(x) + ug(x) \\ \dot{u} = \frac{g(x)a'(x) - f(x)b'(x)}{2r} \\ x(0) = x_0 \\ u(T) = \frac{-b(x(T)) - g(x(T))\Psi'(x(T))}{2r} \end{cases} \quad (27)$$

Corollary 3.7. *We may express u as a function of x (the closed loop control) or x as a function of u by solving the differential equation:*

$$\frac{dx}{du} = \alpha(x) + u\beta(x) \quad (28)$$

where:

$$\alpha(x) = 2r \frac{f(x)}{g(x)a'(x) - f(x)b'(x)} \quad \beta(x) = 2r \frac{g(x)}{g(x)a'(x) - f(x)b'(x)}$$

This corollary is interesting because for certain values of $a(x), b(x), f(x)$ and $g(x)$, we can find a closed form solution for the control.

Note further, $g(x)f'(x) - f(x)g'(x) = 0$ is true if and only if $Ag(x) = f(x)$ for some $A \in \mathbb{R}$. Thus, the optimal control problem structure of interest is correctly stated as:

$$P = \begin{cases} \max \quad \Psi(x(T)) + \int_0^T a(x) + b(x)u + ru^2 \, dt \\ \text{s.t.} \quad \dot{x} = f(x)(1 + Au) \\ x(0) = x_0 \end{cases} \quad (29)$$

Noting that our digital distribution model follows this general form, we move on to examine the case where marginal profit is quality-independent.

3.3 Constant Revenue Case

In Section 3.1, we considered the case when $R(u) = \alpha u$. For many digital products, the price is fixed by the market, rather than by the creator (e.g., most songs on iTunes are \$1.29, most new DVDs are \$14.99 etc.). By framing the revenue stream as quality-independent, the insights from the model change significantly. This assumption is also applicable to markets where consumers are price-sensitive and demand is relatively elastic. In these markets, digital products are similar in price, and brand differentiation is not crucial to success. It is certainly true that quality is important to consumers, but there are many markets where an established price is widely accepted, and the quality focus of a product mostly impacts market share. Below we reframe the optimal control problem with $R(u) = \alpha$.

$$\begin{cases} \max \quad \rho x(T) + \int_0^T \alpha x - Cu^2 \, dt \\ \text{s.t.} \quad \dot{x} = \beta \cdot x(1 - x) \cdot (Pu - \pi_0) \\ x(0) = x_0 \\ u \in [0, 1] \end{cases} \quad (30)$$

Similar to Equation 4, we see that:

$$\begin{cases} \dot{x} = \beta \cdot x(1-x) \cdot (Pu - \pi_0) \\ \dot{u} = -\frac{\alpha\beta P}{2C} \cdot x(1-x) \end{cases} \quad (31)$$

Solving for $\frac{dx}{du}$ yields:

$$\frac{dx}{du} = -\frac{2C}{\alpha P} (Pu - \pi_0) \quad (32)$$

Thus,

$$x = \tilde{C} - \frac{C}{\alpha} u^2 + \frac{2C\pi_0}{\alpha P N} u \quad (33)$$

and

$$u(T) = \left[\frac{\beta P x(T)(1-x(T))\Psi'(x(T))}{2C} \right]_0^1 \quad (34)$$

and $x(u)$ is still maximized when $u = \pi_0/P$. Notice the marginal profit to the revenue stream disappears from Equation 34, when compared to Equation 16 since revenue is quality-independent in this setting. Note, from Expression 33, we can see that Proposition 3.2 also holds in this case and furthermore, simple analysis following the proof of Proposition 3.2 shows that it also holds for the constant revenue case.

The following proposition can be proven by straight forward observation of Equation 34 and the fact that $x(t) \in [0, 1]$:

Proposition 3.8. *The final control value $u(T)$ is always non-negative. If $\frac{\beta P \rho}{2C} < 1$, then $u(T) < 1$.*

If we assume that salvage value of market share is 0, $u(T)$ necessarily goes to 0 as we approach T. Next, we numerically illustrate the relationships between optimal path for focus on quality, market share, and revenue stream for quality dependent revenue as well as constant revenue functional forms.

4 Numerical Examples

We rely on standard techniques in optimal control to derive additional insight into the behavior of x_t and u_t through numerical evaluation. This is particularly relevant when we add a discounting interest rate to the model (see Expression 35), which is not handled in our theoretical analysis. However, for small discount factors (i.e., $r \ll 1$), qualitatively similar behavior will be observed since the discount acts as a small perturbation on the differential equations. Because both market share and focus on quality are dynamic through time, solving for the optimal path of u_t is crucial to understanding the qualitative behavior of the model. Our analysis in this section uses the standard Euler-Lagrange two-point boundary value problem approach.

4.1 Quality Dependent Revenue: Numerical Example

We employ the same system dynamics as Equation 2, so our equation of motion is:

$$\dot{x} = \beta \cdot x_t(1-x_t) \cdot (\pi(u_t) - \pi_0)$$

The profit function is still given as:

$$F = \alpha \cdot u_t \cdot x_t - C \cdot u_t^2$$

Again, this functional states that the revenue stream of the firm depends³ on both u_t and x_t , while costs are quadratic and depend only on u_t . We assume $\Psi(x(T)) = 0$, or that there is no salvage value for market share for products at the end of their life cycle. While this assumption does not always hold in practice, it simplifies the model's insights and does not qualitatively change our results. Introducing a positive value for $\Psi(x(T))$ acts as an upward shift on the optimal path. Assuming a discount rate $0 < r \ll 1$, the single vendor faces the following optimal control problem:

$$\begin{cases} \max \int_0^T e^{-rt} [\alpha \cdot u_t \cdot x_t - C \cdot u_t^2] dt \\ s.t. \dot{x} = \beta \cdot x_t(1 - x_t) \cdot (\pi(u_t) - \pi_0) \\ u_t \in [0, 1], x_t \in [0, 1], x(0) = x_0 \end{cases} \quad (35)$$

As in Section 3.1, we assume $\pi(u_t)$ to be linearly increasing in u_t with the form $\pi(u_t) = P \cdot u_t$. Since our functional and equation of motion satisfy the necessary conditions for a feasible control path, the Hamiltonian remains the same as in equation 5:

$$H(x, u, \lambda, t) = \alpha \cdot u_t \cdot x_t - C \cdot u_t^2 + \lambda_t [\beta \cdot x_t \cdot (1 - x_t) \cdot (P \cdot u_t - \pi_0)]$$

Optimizing with respect to u_t yields the same result as equation 6:

$$u_t = \frac{x_t(-\alpha - \lambda_t \beta P + \lambda_t \beta x P)}{2C}$$

Substituting the solution for u_t back into the equation of motion yields the first differential equation:

$$\dot{x} = \frac{\beta x_t(x_t - 1)(-P x_t \alpha - P^2 x_t \lambda_t \beta + P^2 x_t^2 \lambda_t \beta + 2\pi_0 C)}{2C} \quad (36)$$

From the Pontryagin conditions, we can solve for $\dot{\lambda} = r \cdot \lambda_t - \frac{\partial H}{\partial x}$, which is omitted here. From equations of \dot{x} and $\dot{\lambda}$ in terms of only x_t and λ_t , we can derive an analytic solution for this system of differential equations. However, the answer is largely uninformative and yields no intuition on the nature of the problem. Therefore, we turn to numerical techniques to gain insight into the single vendor's optimal control problem. We use Maple to determine a numerical solution with the following parameters⁴:

$$T = 4; \alpha = 2; \pi_0 = 0.5; P = 1; \beta = 0.5; r = 0.05; C = 1$$

We divide the figures into the the three cases presented in Proposition 3.2. Each case represents a different starting market share; that is to say, the optimal control path for the focus on quality heavily depends on x_0 . Recognizing the initial market conditions is crucial to employing our optimal control model.

In Figure 1, we see the optimal control path for focus on quality and market share for the firm under Case 1 of Proposition 3.2. At this level of market share, it is profitable to maintain a relatively high level of focus on quality to maximize revenue. Market share is monotonically increasing for the product life cycle, even though focus on quality is decreasing, consistent with Lehman's 7th

³The assumption that revenue depends on u_t is again a crucial one.

⁴The parameter T has a small value to ensure convergence of the differential system.

law. Case 1 represents a vendor with a large, established market share. With a strong focus on quality, the firm is able to secure more market share through advertising and word of mouth.

In Figure 2, we see the optimal control path for focus on quality and market share for the firm under Case 3 of Proposition 3.2. Because numerical examples are sensitive to initial values, a decrease in x_0 of 10% is sufficient to shift both the optimal control path and market share behavior. There is less focus on quality at every time t , and market share first increases, peaks, and decreases. Unlike in the first case, the vendor is not incentivized to maintain strong quality dedication.

In Figure 3, we see the optimal control path for focus on quality and market share for the firm under Case 2 of Proposition 3.2. When $x_0 = 0.3$, it is no longer profitable to maintain a high focus on quality; as a result, market share monotonically decreases for the product life cycle. Case 3 represents a vendor with small market share who wishes to maximize revenue without incurring high costs. In our model, it is too costly for this firm to gain market share. Instead, the firm capitalizes on short-term profits before market share deteriorates.

Note, we have set $\rho = 0$, meaning we are studying firms who are not interested in maximizing marketshare. We can take the opposite view of a firm interested solely in maximizing market share at the expense of profit. Such firms exist when they are recently capitalized (i.e., by venture capital). In this case, the optimal control can only be $u(t) = 1$ (for all $t \in [0, T]$) and the result is a (rapid) increase in market share with a corresponding increase in revenue but a (potentially) sub-optimal profit function. As the company increases its marketshare and transitions to Case 1 (see Figure 1), profit becomes a larger motivating factor and marketshare may continue to increase as a result of viral propagation. Ideally (from the company's point of view) this occurs before initial capital is completely expended.

Because conditions are unique for each good and market, market share for an digital good may follow any of the trajectories presented in Figures 1 through 3. If a vendor can identify its marginal benefit and cost of quality dedication, an abstract concept to begin with, the firm can estimate its optimal path to maximize revenue. When firms can estimate values for such as C and x_0 , these numerical examples highlight how initial market share impacts a firm's strategic choice of quality dedication. One interpretation of these numerical results is that having an established reputation and preexisting market share encourages a firm to invest more initial and continual effort into quality. By doing so, the firm can capture even greater market share throughout a product's life cycle. On the other end of the spectrum, a vendor with small market share wants to maintain relatively low quality dedication. These products may be perceived as lower tiered digital goods, and indeed the firm is less focused on their quality by comparison. This self-fulfilling prophecy could add to the discussion of reputation and perceived-quality.

Given the assumptions of our functional form, the three cases present the only possible optimal control paths. The figures are meant to demonstrate the cases in Proposition 3.2, which characterizes a wide class of optimal control problems. Changing the specific parameters or assumptions governing the model would certainly alter the insights from our numerical evaluations. Next, we examine a firm's optimal control path for quality in the constant revenue case when profits are not directly tied to u_t .

4.2 Constant Revenue: Numerical Example

From Section 4.1, we modify the profit function so that revenue is quality-independent. Costs are still quadratic in u_t , and we maintain the general form of Equation 19:

$$F = \alpha \cdot x_t N - C \cdot u_t^2$$

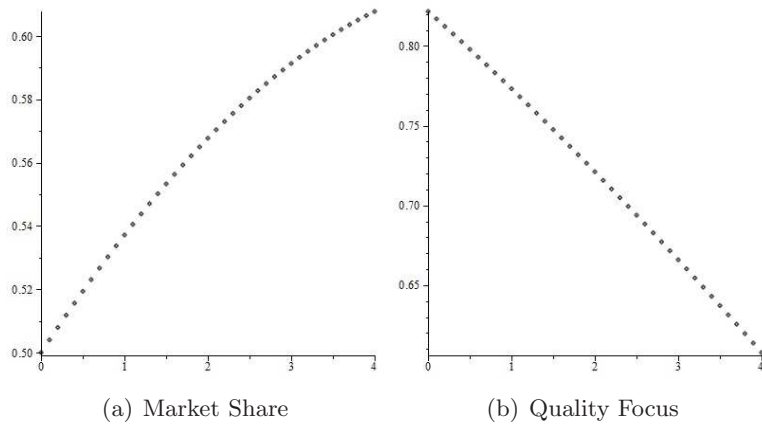


Figure 1: Market conditions with $x_0 = 0.5$

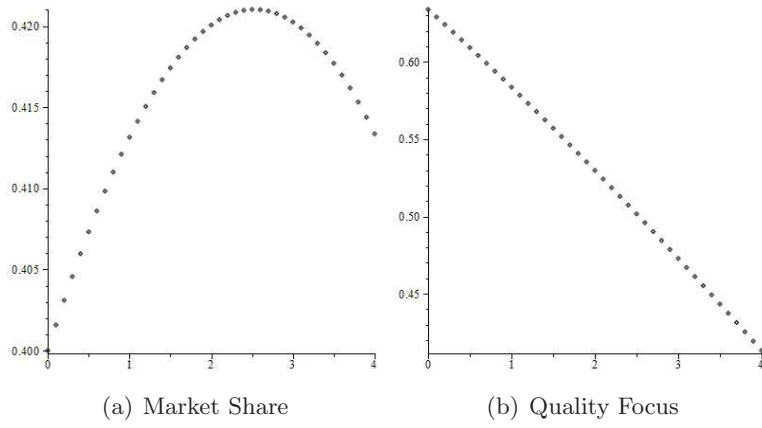


Figure 2: Market conditions with $x_0 = 0.4$

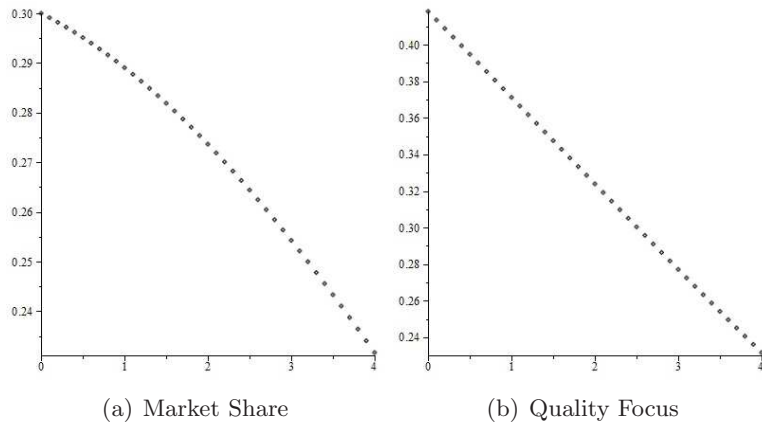


Figure 3: Market conditions with $x_0 = 0.3$

The resulting maximization problem becomes:

$$\begin{cases} \max \int_0^T e^{-rt} [\alpha \cdot Nx_t - C \cdot u_t^2] dx \\ \text{s.t. } \dot{x} = \beta \cdot x_t(1 - x_t) \cdot (\pi(u_t) - \pi_0) \\ u_t \in [0, 1], x_t \in [0, 1], x(0) = x_0 \end{cases} \quad (37)$$

Maintaining the same equation of motion, the quality-independent model fits the generalized model from Section 3.2.

Our Hamiltonian and Euler-Lagrange are modified slightly for this case:

$$u_t = \frac{\beta x_t P \lambda (x - 1)}{2C} \quad (38)$$

and

$$\dot{x} = \frac{\beta x_t (x_t - 1) (\beta P^2 \lambda_t x_t^2 - \beta P^2 \lambda_t x_t + 2\pi_0 C)}{2C} \quad (39)$$

Using the same techniques in Section 4.1, we can solve for $\dot{\lambda} = r \cdot \lambda_t - \frac{\partial H}{\partial x}$ and derive an analytic solution for this system of differential equations, which is largely uninformative. We again use numerical techniques to gain insight into the single vendor's optimal control problem. We hold most of the variables to be the same as before, though this time we vary the cost of quality dedication, C , and hold x_0 constant at 0.5:

$$T = 4; \alpha = 2; \pi_0 = 0.5; P = 1; \beta = 0.5; r = 0.05; x_0 = 0.5$$

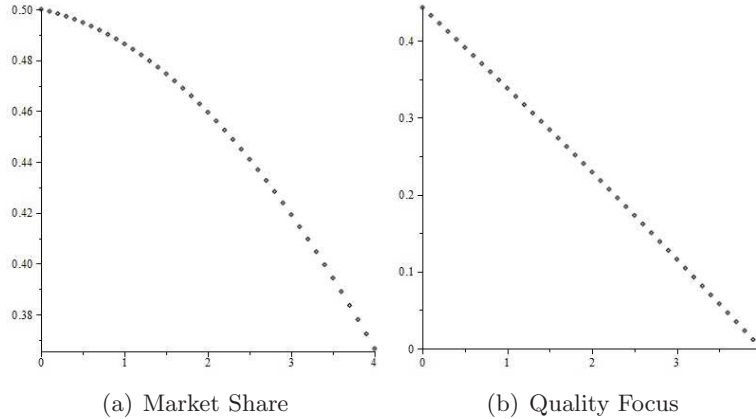


Figure 4: Market conditions with $C = 1$

In Figure 4, we can see the implications of market conditions where focus on quality does not directly impact revenue stream. All parameters are the same in Figure 1 and 4, but a firm has less incentive to focus on quality in the second setting. This leads to a lower u_t at every point and decreasing market share instead of increasing market share. In Figure 5, C is set to 0.5, decreasing the cost of quality dedication. This illustrates that firms may still wish to focus on quality, even when marginal revenue is not increased, in an effort to gain market share. Note that in both scenarios, u_t necessarily decreases to 0 at time T for optimality; this is a consequence of setting salvage value to 0, but it does not qualitatively change insights.

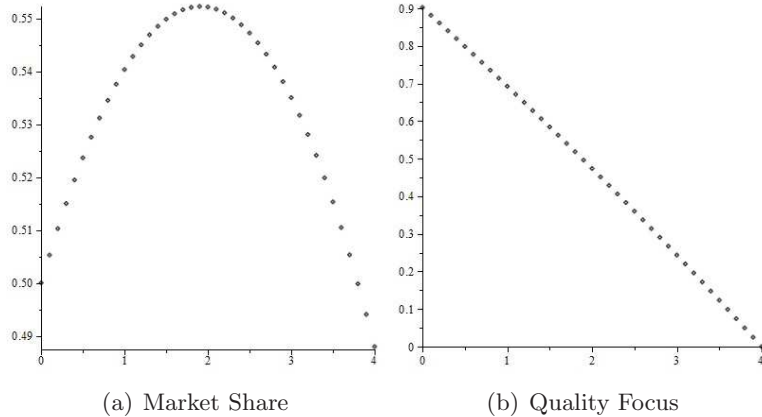


Figure 5: Market conditions with $C = 0.5$

Our model prescribes different optimal control paths for focus on quality depending on the market setting. Factors such as initial market share, cost of effort towards quality, and revenue characteristics of the objective function can all change the optimal behavior of firms. In this section, we highlight the cases outlined in Proposition 3.2, as well as optimal behavior assuming quality-independent revenue.

5 Conclusion

In this paper, we have proposed an optimal control problem to model a firm’s strategic decision-making process during mobile app development and distribution. We first examine the relationship between market share and focus on quality using a rumor spreading dynamic. Our theoretical analysis demonstrates that market share behavior can be characterized as a function of quality dedication and initial conditions; thus, we generate closed-form differential equations that provide insight into the fundamental relationship between market share and quality dedication. These insights can be generalized to other optimal control problems that share a similar structure.

To help visualize our findings, we employ numerical examples that emphasize the different optimal control paths under different market conditions. As expected, initial market share and cost of quality dedication play critical roles in the vendor’s decision-making process. These findings are consistent with our theoretical propositions and software evolution theory on quality. We look to extend our research by analyzing the case with multiple vendors in a digital distribution setting. The addition of other firms changes the dynamics of the model from an optimal control problem to a differential game, as all players must account for the actions of all other players at every time t . We hope to better understand how our findings hold up in the robust setting.

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